EDU SCHOOL
ijpsrs.com
INTERNATIONAL JOURNAL OF PURE SCIENCE RESEARCH STUDIES (IJPSRS)
(Open Access, Double-Blind Peer Reviewed Journal)
ISSN Online: ISSN Print

Random Walks On Graphs And expected Hitting Times

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Article Information

Received: 12th November 2025
Received in revised form: 15th December 2025
Accepted: 16th January 2025
Available online: 14th February 2026

Volume: 2
Issue: 1
DOI: https://doi.org/10.5281/zenodo.18679402

Abstract

Random walks on graphs constitute a fundamental framework in probability theory and combinatorics with extensive applications across mathematics, computer science, and physics. This paper examines the theoretical foundations of random walks on finite graphs with emphasis on expected hitting times, which quantify the average number of steps required for a random walk to reach a target vertex from a given source. We establish the connection between random walks and Markov chains, derive the fundamental system of linear equations governing hitting times, and present closed-form solutions for several canonical graph structures including paths, cycles, and complete graphs. The analysis employs techniques from linear algebra, particularly the properties of the transition matrix and its relationship to the graph Laplacian. We demonstrate applications of hitting time calculations to problems in network analysis, algorithm design, and theoretical computer science, providing both theoretical results and computational examples. The methods presented offer insights into the structural properties of graphs and their influence on stochastic processes.

Keywords: Random Walk, Markov Chain, Hitting Time, Graph Theory, Transition Matrix, Stationary Distribution.

1. INTRODUCTION

Random walks on graphs represent one of the most extensively studied stochastic processes in mathematics, with roots extending to early probability theory and modern applications spanning numerous disciplines. A random walk on a graph G = (V, E) is a stochastic process where a particle moves from vertex to vertex along the edges of the graph, with transition probabilities determined by the graph structure. At each time step, the walker at vertex u moves to a neighboring vertex v with probability proportional to the edge weight connecting them, or with equal probability in the case of unweighted graphs.[1]

The concept of expected hitting time is central to understanding the temporal dynamics of random walks. For vertices s and t in a graph, the expected hitting time H(s, t) is defined as the expected number of steps for a random walk starting at vertex s to first reach vertex t. This quantity encodes important structural information about the graph and has proven instrumental in analyzing network properties, designing randomized algorithms, and understanding physical phenomena such as diffusion processes.[2]

The mathematical framework for analyzing random walks on graphs draws heavily from the theory of Markov chains. A random walk on a graph naturally defines a discrete-time Markov chain where the states correspond to vertices and transition probabilities are determined by the edge structure. This connection enables the application of powerful techniques from probability theory and linear algebra to derive fundamental results about hitting times, commute times, cover times, and mixing times.[3]

This paper is organized as follows. We begin by establishing the theoretical framework, defining random walks formally within the context of Markov chains and introducing key notation and concepts. We then develop the mathematical theory of expected hitting times, deriving the fundamental system of linear equations and examining solution techniques. Subsequent sections present explicit calculations for specific graph families and discuss applications in computer science and network analysis. Throughout, we emphasize both theoretical rigor and computational tractability.

2. THEORETICAL FRAMEWORK

2.1. Graphs and Random Walks

Let G = (V, E) be a finite, connected, undirected graph where V represents the vertex set with |V| = n vertices and E represents the edge set. For vertices u and v, we denote by d(u) the degree of vertex u, defined as the number of edges incident to u. In an unweighted graph, the transition probability from vertex u to vertex v is given by

$$P(u,v) = \frac{1}{d(u)} \quad \text{if } (u,v) \in E \tag{1}$$

$$P(u,v) = 0 \qquad \text{if } (u,v) \notin E$$

This defines a stochastic process where at each time step t, if the walker is at vertex u, it moves to a uniformly random neighbor of u. The collection of all transition probabilities forms the transition matrix P, an n × n stochastic matrix with rows summing to one.

2.2. Markov Chain Formulation

A random walk on a graph constitutes a discrete-time Markov chain with state space V. The Markov property ensures that the future evolution of the walk depends only on the current position, not on the history of previously visited vertices. Formally, for the stochastic process $\{x_t\}t \geq 0$ representing the position of the walker at time t, we have

$$P\left(X_{t+1} = v | X_t = u, X_{t-1} = w_{t-1,\dots,} X_0 = w_0\right) = P(X_{t+1} = v | X_t = u) = P(u,v) \tag{2}$$

For connected, non-bipartite graphs, the random walk is irreducible and aperiodic, guaranteeing the existence of a unique stationary distribution π satisfying πP = π. For simple random walks on undirected graphs, the stationary probability of being at vertex v is proportional to its degree. [4]

$$\pi(v) = \frac{d(v)}{2|E|} \tag{3}$$

where 2|E| equals the sum of all vertex degrees. This stationary distribution plays a crucial role in analyzing long-term behavior and computing various random walk metrics.

Fig 1: Random Walk on an Undirected Graph

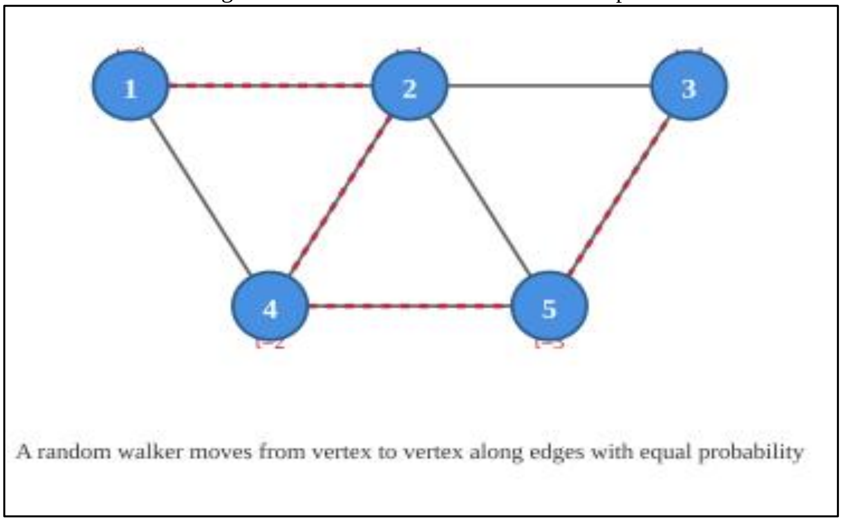

Figure 1. Illustration of a random walk on an undirected graph. The walker moves from vertex to vertex along edges, with each step chosen uniformly at random from the available neighbors. The red dashed path shows a sample trajectory from t = 0 to t = 4

3. EXPECTED HITTING TIMES: MATHEMATICAL THEORY

3.1. Definition and Basic Properties

For vertices s and t in graph G, we define the hitting time H(s, t) as the expected number of steps for a random walk starting at vertex s to reach vertex t for the first time. Formally, if T_t denotes the first time the walk visits vertex t, then

$$H(s,t) = E_s[T_t] = E[T_t \mid X_0 = s] \quad (4)$$

where E_s denotes expectation conditioned on starting at vertex s. The hitting time satisfies several fundamental properties. First, by definition, H(t, t) = 0 for any vertex t. Second, the hitting time from s to t need not equal the hitting time from t to s in general graphs, though symmetry holds in specific structures such as complete graphs.

3.2. Fundamental Equations for Hitting Times

The expected hitting times satisfy a system of linear equations derived from first-step analysis. For s ≠ t, conditioning on the first step of the walk yields

$$H(s,t) = 1 + \sum_{v \in N(s)} P(s,v) H(v,t) \quad (5)$$

where N(s) denotes the set of neighbors of vertex s. This equation expresses that the expected hitting time from s to t equals one step plus the weighted average of hitting times from the neighbors of s to t, weighted by the transition probabilities. Combined with the boundary condition H(t, t) = 0, this yields a system of n - 1 linear equations in n - 1 unknowns.[5]

For simple random walks on regular graphs where d(v) = d for all vertices v, the equation simplifies to

$$H(s,t) = 1 + \frac{1}{d} \sum_{v \in N(s)} H(v,t) \quad (6)$$

3.3. Matrix Formulation and Solution Methods

The system of equations for hitting times can be expressed in matrix form. Let h_t denote the vector of hitting times to target vertex t from all other vertices. Removing the row and column corresponding to t from the transition matrix P yields the reduced matrix $\tilde{P}$. The hitting time vector satisfies

$$h_t = 1 + \tilde{P} h_t \quad (7)$$

where 1 denotes the all-ones vector. Rearranging gives

$$(1 - \tilde{P}) h_t = 1 \quad (8)$$

Which has the unique solution $h_t = (I - \overline{P})^{-1} 1$, provided the matrix I - $\tilde{P}$ is invertible. For connected graphs, this matrix is indeed invertible, establishing the existence and uniqueness of hitting times.[6]

Fig 2: Conceptual illustration of expected hitting time.

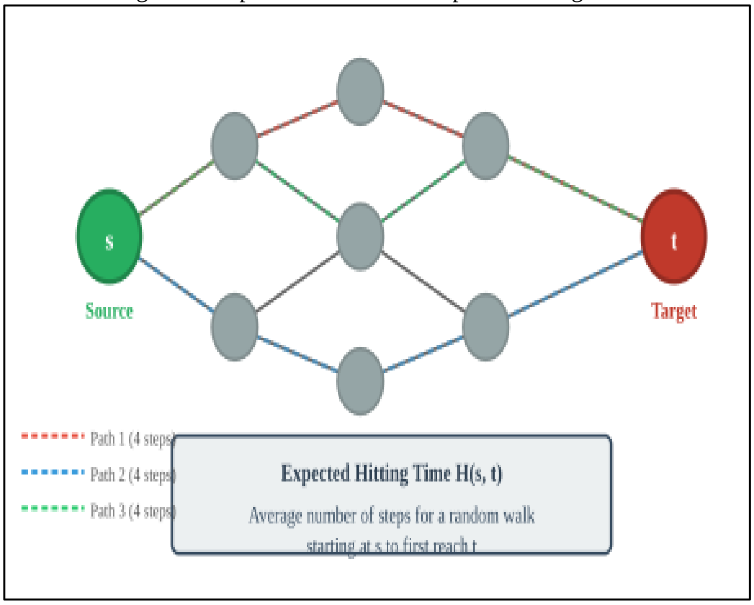

Figure 2. Conceptual illustration of expected hitting time. Multiple possible paths exist from source vertex s to target vertex t, each with different lengths. The expected hitting time H(s, t) represents the average path length over all possible random walk trajectories.

4. HITTING TIMES ON SPECIFIC GRAPH STRUCTURES

4.1. Path Graphs

Consider a path graph P_n with n vertices labeled 1, 2, ..., n arranged linearly. For 1 < i < n, vertex i has neighbors i - 1 and i + 1, while vertices 1 and n have degree one. We derive the hitting time from vertex 1 to vertex n.

Let $h_i = H(i_1, \eta)$ denote the expected hitting time from vertex i to vertex n. The boundary condition gives $h_n = 0$. For interior vertices i with 1 < i < n, the first-step equation yields

$$h_i = 1 + \frac{1}{2}h_{i-1} + \frac{1}{2}h_{i+1} \quad (8)$$

Rearranging gives $h_{i+1} - 2h_i + h_{i-1} = -2$, a second-order linear difference equation. For vertex 1, we have $h_1 = 1 + h_2$. Solving this system yields the quadratic solution

$$H(i, n) = (n - i)(n + i - 1) \quad (9)$$

In particular, H(1, n) = (n - 1)n, demonstrating that the expected hitting time from one end of a path to the other grows quadratically with path length.[1]

4.2. Cycle Graphs

A cycle graph C_

n consists of n vertices arranged in a circle, with each vertex having degree two. By symmetry, the hitting time from vertex 1 to vertex k depends only on the distance d = min(k - 1, n - k + 1) between them along the cycle. The expected hitting time formula is

$$H(1, k) = d(n - d) \quad (10)$$

where d represents the shorter arc distance. For the maximum distance $d = \frac{n}{2}$ (when n is even), this yields $H\left(1, \frac{n}{2} + 1\right) = \frac{n^2}{4}$, again demonstrating quadratic growth with graph size.

4.3. Complete Graphs

The complete graph K_n has all possible edges between n vertices. Every vertex connects to every other vertex, making the graph highly symmetric. For any two distinct vertices s and t, the transition probability from s to t is $\frac{1}{n-1}$. By symmetry, H(s, t) is the same for all pairs of distinct vertices.
The first-step analysis gives $h = 1 + \frac{n-2}{n-1}h$ for the common hitting time h. Solving yields

$$H(s, t) = n - 1 \text{ for all } s \neq t \text{ in } K_n \quad (11)$$

This linear dependence on n contrasts sharply with the quadratic growth observed in paths and cycles, illustrating how graph connectivity dramatically affects hitting time behavior.[2]

Fig 3: Common Graph Structures for Random Walk Analysis

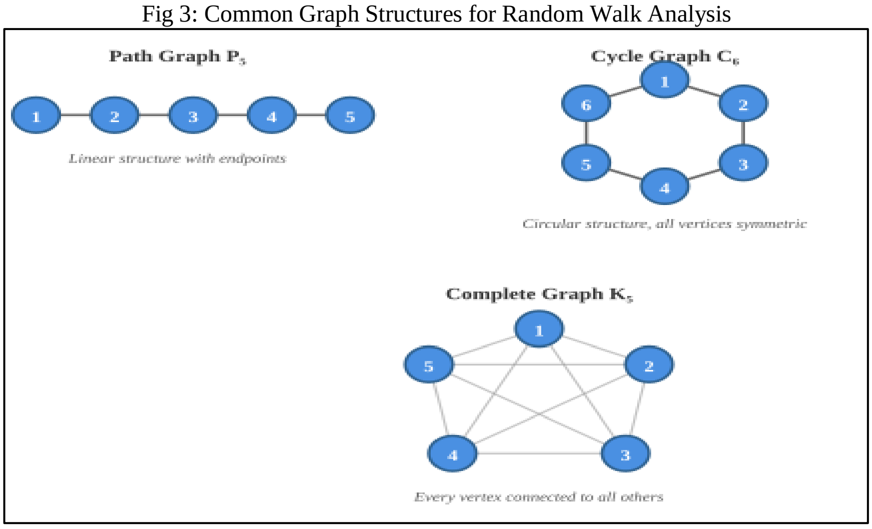

Figure 3. Three fundamental graph structures with distinct hitting time characteristics. Path graphs exhibit quadratic growth, cycle graphs show distance-dependent behavior, and complete graphs demonstrate linear scaling with number of vertices.

Table 1. Expected hitting times for canonical graph structures

Graph Type	**Vertices**	**Maximum H(s, t)**
Path P_n	s = 1, t = n	n(n - 1)
Cycle C_n	Adjacent vertices	$n^2/4$ (n even)
Complete K_n	Any s ≠ t	n – 1

Note**.** Hitting times scale quadratically for paths and cycles, but linearly for complete graphs, demonstrating the impact of graph connectivity on random walk dynamics.

5. APPLICATIONS AND COMPUTATIONAL METHODS

Expected hitting times find extensive application across multiple domains. In computer science, they inform the analysis of randomized algorithms, particularly in distributed computing and network protocols. The cover time of a graph, defined as the expected time to visit all vertices, directly relates to hitting times and provides performance guarantees for graph exploration algorithms.[7]

In network analysis, hitting times quantify the accessibility between nodes in communication networks, social networks, and biological networks. The commute time $C(s,t) = \dot{H}(s,t) + H(t,s)$ provides a symmetric distance metric on graphs that accounts for the graph structure. This metric has applications in graph clustering, where vertices with small commute times are grouped together.

Computational methods for hitting times include direct linear algebra approaches and Monte Carlo simulation. For small graphs, solving the system $(I - \bar{P})h_t = 1$ via Gaussian elimination or matrix inversion yields exact results. For large networks, iterative methods such as successive over-relaxation or conjugate gradient provide efficient approximations. Alternatively, simulating many random walks and averaging their hitting times produces empirical estimates with confidence intervals.[4]

Recent developments connect hitting times to spectral graph theory through the relationship between random walks and the graph Laplacian matrix L = D - A, where D is the degree matrix and A is the adjacency matrix. The eigenvalues and eigenvectors of the normalized Laplacian encode information about mixing times and commute distances, providing powerful tools for analyzing random walk behavior on complex networks.

6. CONCLUSION

This paper has examined the mathematical framework of random walks on graphs with particular emphasis on expected hitting times. We established the connection between random walks and Markov chains, derived the fundamental system of linear equations characterizing hitting times, and computed explicit formulas for canonical graph structures including paths, cycles, and complete graphs. The analysis demonstrates how graph topology profoundly influences the temporal behavior of random walks, with hitting times ranging from linear to quadratic growth depending on connectivity.

The theoretical results presented provide a foundation for understanding stochastic processes on networks and offer practical tools for analyzing real-world systems. Applications span algorithm analysis, network design, and physical modeling. Future research directions include extending these methods to weighted graphs, directed graphs, and continuous-time random walks, as well as investigating the relationship between hitting times and other graph parameters such as resistance distance and mixing times.

The interplay between combinatorial structure and probabilistic behavior remains a rich area of mathematical investigation, with random walk theory serving as a fundamental bridge between graph theory, probability, and computational mathematics. As networks grow increasingly central to science and technology, the mathematical principles governing random walks and hitting times will continue to provide essential insights into complex systems.

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